



# A New Multi-dimensional Transform for Digital Signal Processing Using Generalized Association Schemes

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## i Introduction

In a large number of scientific areas, one needs to store data (e.g. signals, images, or statistical samples) in the form of arrays. In the classical approaches one considers two-dimensional arrays only which are usually referred to as the *contingency tables* or simply as *matrices*. A number of properties of the data could be investigated in terms of the structure of the linear algebra generated by these matrices. However, one could also store data instead in the form of multi-dimensional arrays, one reason for doing this lies in storing the large data in a compact way. This kind of representation of data has been found to be very useful in many signal processing applications especially in the processing of image data such as satellite reconnaissance photographs, medical imagery including X-ray images, seismic records, and electron micrographs. Consequently, there has been considerable interest in the recent past to develop fast algorithms for computing multi-dimensional transforms

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and convolutions, [2], [7]. Besides these, there are other applications of multi-dimensional arrays in some areas such as the design of experiments, transportation planning and defense strategic planning [13]. Further, in the design of VLSI structures, multi-dimensional arrays play a significant role. For example, some companies have designed separately 3-D microelectronic structures but several major problems need to be overcome before such 3-D architectures become commercially available.

In all the work done so far to develop the theory of multi-dimensional arrays, there has been no attempt to define a *multiplication* operator, the operators which are in current use are the straightforward extensions of the usual definitions of matrix addition and scalar multiplication. It is thus natural to think of constructing a higher dimensional analog of the concept of matrix multiplication and explore its applications.

Our initial interest in the study of multi-dimensional arrays came from our recent work on combinatorics [11]. The concept of *association schemes* is well-known in combinatorics with applications in coding theory and design of experiments. Corresponding to an association scheme, there is an associative algebra, called the *Bose-Mesner algebra* which is instrumental in studying the properties of association schemes. Several people have studied extensively the properties of the Bose-Mesner algebra from the point of view of combinatorics and algebraic coding theory; the theory now appears in [1], [6], [10]. In [41], we defined a combinatorial structure called an **association scheme on triples**, in short, an *AST* which is a higher dimensional analog of the association schemes and also developed a **non-associative ternary algebra** which is a generalization of the Bose Mesner algebra. Several examples of *AST*'s have been constructed by us using 2-designs, 2-transitive permutation groups and two graphs; some of our constructions give infinite families of *AST*'s. One motivation of ours for extending the concept of association schemes to higher dimensions was to formulate the concept of block design which is *partially balanced* with respect to a 3-subset. The ternary algebra has been investigated further by us in [12].

There are some interesting applications of multi-dimensional arrays in digital signal processing, control theory, statistics and biological sciences (see for example, [3]-[4]; [9], [13]). However, in none of these works any algebra (ternary or n-ary) has been defined.

## NOTATION

For any  $m \times n$  matrix  $X$ , the  $(i, j)$  entry is denoted by  $(X)_{ij}$ .

In what follows, a three-dimensional array of elements from any field will be called a *three-dimensional matrix*, or often, briefly, a *matrix*. (It will be clear from the context whether the *matrix* is 2-dimensional or 3-dimensional).

If  $A$  is any  $m \times n \times q$  matrix and  $s, t$  are fixed integers,  $1 \leq s \leq m$  and  $1 \leq t \leq n$ , then

$(A)_{ijk}$  denotes the  $(i, j, k)$  entry of  $A$ .

$A_{s..}$  denotes the  $n \times q$  matrix whose  $(j, k)$  entry is  $(A)_{s,j,k}$ .

$A_{.st}$  denotes the  $q$ -dimensional vector whose  $k^{th}$  component is  $A_{stk}$ . Similarly, one defines matrices  $A_{.s.}$ ,  $A_{..s}$ ,  $A_{.st}$  etc by changing the ranges of  $s$  and  $t$  suitably.

(The  $n \times q$  matrices  $A_{s..}$ ,  $1 \leq s \leq m$  represent the  $m$  vertical plane-sections of the three-dimensional matrix  $A$ . This also gives a convenient method to list the elements of the matrix).

If  $\{D^{ab} : 1 \leq a \leq u, 1 \leq b \leq v\}$  is a family of matrices of size  $m \times n$  each, then  $(D^{..})_{ij}$  denotes the  $u \times v$  matrix whose  $(p, q)$  entry is  $(D^{pq})_{ij}$ .

## 2 Transform

### 2.1 Background

A *discrete transform* of an  $m \times n$  matrix  $A$  is defined (for example, in Rosenfeld [14, p.21]) as the product

$$F = PAQ \quad (1)$$

where  $P$  and  $Q$  are certain given nonsingular  $m \times m$  and  $n \times n$  matrices respectively. Thus, for  $1 \leq i \leq m, 1 \leq j \leq n$ ,

$$(F)_{ij} = \sum_{r=1}^m \sum_{s=1}^n (P)_{ir} (A)_{rs} (Q)_{sj} \quad (2)$$

Since  $P$  and  $Q$  are nonsingular, one can compute the *inverse transform* of  $F$ . By specializing  $P$  and  $Q$  one obtains a number of transforms, the most well known among them is the *discrete Fourier Transform* which is obtained by choosing nonsingular matrices  $P$  and  $Q$  in (1) such that

$$(P)_{ij} = \frac{1}{m} \exp\left(\frac{2\pi i j \sqrt{-1}}{m}\right), \quad (Q)_{rs} = \frac{1}{n} \exp\left(\frac{2\pi r s \sqrt{-1}}{n}\right) \quad (3)$$

where  $1 \leq i, j \leq m$  and  $1 \leq r, s \leq n$ . The Fourier transform has found a very wide range of practical applications (for a survey of its properties and also a number of other widely used transforms such as the *Hadamard Transform*, *Sine* and *Cosine* Transforms, etc. see, for example, [7]). The development of a *fast* algorithm by Cooley and Tukey in 1965 to compute the Fourier transform has revolutionized a number of areas such as medical diagnostics using the CAT scan (see for example, Blahut [2] for a survey of Cooley and Tukey's theory and later developments).

In the definition of the discrete transform given by (1), if  $P, A, Q$  are all  $n \times n$  matrices, then the total number of multiplications needed to compute  $F$  is  $O(n^4)$ . For, in (2) consider a fixed  $(i, j)$ . In order to compute  $F_{ij}$  one performs  $n^2$  multiplications corresponding to the  $n^2$  terms in the double summation, assuming that all the products of the form  $(P)_{ir}$  and  $(Q)_{sj}$  have been *pre-computed* and stored for use whenever required. As there are  $n^2$  choices of  $(i, j)$ , the total number of multiplications is  $O(n^4)$ .

Let  $A, B, C$  be 3-dimensional matrices such that  $A$  is of size  $m \times n \times q$ ,  $B$  is of size  $p \times m \times q$  and  $C$  is of size  $p \times n \times m$ . Then the **ternary product**  $Y = ABC$  is a matrix of size  $p \times n \times q$  defined by

$$(Y)_{ijk} = \sum_{h=1}^m (A)_{hjk} (B)_{ihk} (C)_{ijh} \quad (4)$$

The ternary product is neither associative nor commutative. However, it is multi-linear. Two pairs of matrices  $(A, B)$  and  $(A', B')$  are called (left) *equivalent* if

$$ABX = A'B'X \quad (5)$$

for every conformable matrix  $X$ . It is easy to see that this is an equivalence relation on pairs of matrices of given dimensions. A pair  $C, D$  is a (left) *inverse pair* of  $A, B$  if

$$CD(ABX) = X \quad (6)$$

for every conformable  $X$ . If  $(C, D)$  is an inverse pair of  $(A, B)$  then so is any pair  $(C', D')$  equivalent to  $(C, D)$ . A pair of matrices  $(A, B)$  may not always have an inverse pair, if it has an inverse pair, then we refer to  $(A, B)$  as an *invertible pair*.

## 2.2 New Transforms

**Definition 2.2.1 A New 3D Transform.** Let  $(P, Q)$  be a pair of invertible matrices of sizes  $m \times n \times q$  and  $p \times n \times q$  respectively. Let  $A$  be a matrix of size  $p \times m \times q$ . Then, the *transform* of  $A$  is the matrix  $B$  of size  $p \times n \times q$  given by

$$(B)_{ijk} = \sum_{h=1}^m (P)_{hjk} (Q)_{ihk} (A)_{ijh} \quad (7)$$

Thus, using the ternary product defined in (3), we have that

$$B = PQA \quad (8)$$

Definition 2.2.1 is quite general in the sense that corresponding to various choices of invertible pairs  $(P, Q)$  one would obtain different transforms of the given matrix. In Section 4 we shall take a number of families of invertible pairs which we have worked out in [12] and use them to obtain specific transforms.

**2.2.2 Some New 2D Transforms.** Using Definition 3.1, we can obtain as special cases two new 2-dimensional transforms: we shall refer to them for convenience as transforms of **Type 1** and **Type 2**; these transforms are both new in the sense that they are distinct from the discrete transform given by (1).

(i) **Type 1 Transform.** Put in (4)  $i = \alpha$  where  $\alpha$  is any (fixed) integer such that  $1 \leq \alpha \leq p$ . Then (4) gives for all  $1 \leq j \leq n, 1 \leq k \leq q$ ,

$$B_{\alpha jk} = \sum_{h=1}^m (P)_{hjk} (Q)_{\alpha hk} (A)_{\alpha jh}$$

Using the notation described in Section 1, this implies that

$$B_{\alpha..} = PQ_{\alpha..} A_{\alpha..} \quad (9)$$

Thus, we have a *new 2-dimensional transform*  $B_{\alpha..}$  of the  $m \times q$  matrix  $A_{\alpha..}$ . (Notice that the matrices  $(P, Q_{\alpha..})$  form an invertible pair since  $(P, Q)$  is an invertible pair.)

(ii) **Type 2 Transform.** To obtain the transform of *Type 2*, fix in (4),  $j = \beta$  where  $1 \leq \beta \leq n$ , getting

$$B_{..\beta} = P_{..\beta} Q A_{..\beta} \quad (10)$$

which gives a transform of  $A_{..\beta}$ . (Again, notice that  $(P_{..\beta}, Q)$  is an invertible pair since  $(P, Q)$  is.)

**2.2.3. A New 1D Transform.** In (4) fix  $i = \alpha, k = \beta$ , say. Then, it follows from (4) that for all  $1 \leq j \leq n$ ,

$$B_{\alpha j \beta} = \sum_{h=1}^m (P)_{h j \beta} (A)_{\alpha h \beta} (Q)_{\alpha j h} \quad (11)$$

and so using the notation described in Section 1, this implies that

$$B_{\alpha\beta} = P_{\beta} A_{\alpha\beta} Q_{\alpha} = (P_{\beta} \odot Q_{\alpha}) A_{\alpha\beta} \quad (12)$$

Notice that the right hand side of equation (7) is the usual matrix product of the *Hadamard product* of the matrices  $P_{\beta}$  and  $Q_{\alpha}$  with the matrix  $A_{\alpha\beta}$ . We can also devise variations of (7). For example, we have the transforms  $B_{\alpha\beta}$  and  $B_{\alpha\beta}$  obtained from (4) in a similar manner.

**Remark 2.2.4.** A *conceptual advantage* enjoyed by the new 2-dimensional transforms defined in 4.2 for an  $n \times n$  matrix over the traditional discrete transform given by (1) can be seen as follows. Any family  $\mathcal{F}$  of  $n$  matrices each of size  $n \times n$  can be represented by an  $n \times n \times n$  array  $A$ , say, where each *vertical layer*  $A_{i..}$ ,  $1 \leq i \leq n$  represents the  $i^{\text{th}}$  matrix of the set  $\mathcal{F}$ . If  $B$  is the transform of  $A$  according to Definition 2.2.1 (corresponding to a given pair of invertible matrices the vertical layers  $B_{i..}$  of  $B$  give the 2-dimensional transforms of Type 1 defined above of the matrices  $A_{i..}$ . Thus, all the matrices of the given family  $\mathcal{F}$  can be transformed *in one step* by computing the 3-dimensional transform of  $A$  using Definition 2.2.1. (Another possible way of representing the matrices of the family  $\mathcal{F}$  is by taking the *horizontal layers* of  $A$  to be the members of  $\mathcal{F}$ . Then, the transform of Type 2 of each member of  $\mathcal{F}$  can be computed from the 3-dimensional transform  $B$  by taking its horizontal layers.) Thus our proposed 3-dimensional transform is aimed at computing collectively the transforms of a family of matrices instead of computing them individually one by one. So, the 3D transform is quite suitable for the computation of 2D transforms (of Type 1 or Type 2) of a large family of matrices.

**Remark 2.2.5** The total number of multiplications needed to compute the discrete transform (7) is  $O(n^4)$ . In computing the 2-dimensional transforms of Type 1 and Type 2 defined in this section, the total number of multiplications for each of these is  $O(n^3)$ . It is well known that the architecture of a computer (sequential or parallel) is such that the operation of multiplication takes much more time to perform as compared to addition.

- Thus there is a significant computational advantage in the transform given by Definition 2.2.1 over the standard discrete transform given by (7) since they involve  $O(n^3)$  and  $O(n^4)$  multiplications respectively.

The following result describes some basic properties of the 3D transform.

**Proposition 2.2.6.** (i) If  $(P, Q)$  and  $(P', Q')$  are two equivalent invertible pairs, then

$$B = PQA, \quad B' = P'Q'A \implies B = B' \quad (13)$$

where the matrices are assumed to be conformable for forming ternary product.

(ii) If  $(R, S)$  is the inverse pair of a given invertible pair  $(P, Q)$ , then

$$B = PQA \implies RSB = A. \quad (14)$$

It follows from 2.2.6 that our proposed transform does have some desirable properties which is expected of a transform; in particular, if  $B$  is the transform of  $A$ , then it is possible to recover  $A$  uniquely from  $B$ .

**Remark 2.2.7.** Let  $B$  be the transform of  $A$  according to Definition 2.2.1. Then for any integers  $\alpha$  and  $\beta$  with  $1 \leq \alpha \leq m$  and  $1 \leq \beta \leq n$ , one has from [12, (6), 4.7] that

$$B_{\alpha\beta} = A_{\alpha\beta} \cdot F^{\beta\alpha}. \quad (15)$$

where

$$F^{\beta\alpha} := P_{\alpha} \odot Q_{\alpha} \quad (16)$$

In (15) the product on the right hand side of the equation is the usual matrix product of 2-dimensional matrices and in (16) the product  $\odot$  is the standard Hadamard product. We have shown [12, 4.10] that the square matrix  $F^{\alpha\beta}$  is invertible if  $(P, Q)$  is an invertible pair: this is indeed the case in the present situation. Now, comparing (15) with the definition of the discrete transform given by (1), one sees that  $B_{\alpha\beta}$  is a discrete transform of the matrix  $A_{\alpha\beta}$ . In other words, there are certain *slices* in  $B$  which are the discrete transforms of certain slices of  $A$  by some matrices which are obtained from the given invertible pair  $(P, Q)$ . For example, if we choose matrix  $F^{\alpha\beta}$  such that

$$F^{\alpha\beta} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^i & \omega^{2i} & \dots & \omega^{(n-1)i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \dots & \dots & \omega^{(n-1)^2} \end{bmatrix} \quad (17)$$

then each  $F^{\alpha\beta}$  obtained by (16) is the **Fourier transform** of  $A_{\alpha\beta}$  for all admissible values of  $\alpha, \beta$ . Thus if we can find invertible pairs  $(P, Q)$  such that some of its slices given by (16) have nice forms (such as some well known 2-dimensional transform like (10)) then the three-dimensional transform thus obtained from Definition 2.2.1 is such that some of its specific slices are the corresponding two-dimensional transforms.

**Remark 2.2.8.** Let  $A_{\alpha\beta}, T_{\alpha\beta}$  be two one-dimensional matrices and  $A_{\alpha\beta}^*, T_{\alpha\beta}^*$  be their Fourier transforms respectively. Let  $C_{\alpha\beta}$  be the cyclic convolution of  $A_{\alpha\beta}$  and  $T_{\alpha\beta}$  and let  $C_{\alpha\beta}^*$  be the Fourier transform of  $C_{\alpha\beta}$ . Then, by the well known *convolution theorem* ([2]) one has that

$$C_{\alpha\beta}^* = A_{\alpha\beta}^* \odot T_{\alpha\beta}^* \quad (18)$$

Now, clearly

$$A_{\alpha\beta}^* = A_{\alpha\beta} \cdot F^{\beta\alpha}, \quad T_{\alpha\beta}^* = B_{\alpha\beta} \cdot F^{\beta\alpha}, \quad C_{\alpha\beta}^* = C_{\alpha\beta} \cdot F^{\beta\alpha} \quad (19)$$

So,

$$C_{\alpha\beta} \cdot F^{\beta\alpha} = (A_{\alpha\beta} \cdot F^{\alpha\beta}) \odot (B_{\alpha\beta} \cdot F^{\alpha\beta}) \quad (20)$$

or,

$$C_{\alpha\beta} = \frac{1}{F^{\beta\alpha}} (A_{\alpha\beta} \cdot F^{\alpha\beta}) \odot (B_{\alpha\beta} \cdot F^{\alpha\beta}) \quad (21)$$

Thus for the proposed three-dimensional transform we see that (21) is an *analog of the well known convolution theorem*.

### 3 Special Cases of Transforms

As remarked in Section 3, different choices of invertible pairs in the Definition 2.2.1 would give different classes of transforms for a given three-dimensional matrix  $A$ . In this section we shall describe two specific choices of invertible pairs arising in a combinatorial setting and obtain the corresponding transforms.

#### 3.1 Background

The ternary product defined in (4) generates an algebra (called a *ternary algebra*) for all matrices with real entries of a given size  $v \times v \times v$ . Such a ternary algebra arises in a natural way from a combinatorial structure proposed by us called an **association schemes on triples (AST)** ([41]). Any AST has a set of *relations*  $R_0, \dots, R_m$ , say, of which the relations  $R_0, \dots, R_4$  are called the *trivial* relations in the sense that they are the same in every AST, the relations  $R_i$  for  $i = 4, \dots, m$  are called the *nontrivial* relations. Now, each  $R_i$  can be represented in a natural way by an *adjacency matrix*  $A_i$ , say, of size  $v \times v \times v$  where  $v$  is the cardinality of the set  $\Omega$  on which the AST is defined. It has been proved by us ([41, Theorem 2.1]) that the 3-dimensional matrices  $A_i$ ,  $0 \leq i \leq m$  generate a ternary algebra. Further, we have proved that the adjacency matrices corresponding to the nontrivial relations in an AST form a *subalgebra* of the full ternary algebra. We have shown ([41]) that a large number of AST's can be constructed (including several infinite families) from combinatorial structures like the projective planes, block designs, and permutation groups: so, we have a rich supply of the corresponding ternary algebras. Interestingly, in a large number of these AST's there are only two nontrivial relations and so the corresponding ternary subalgebra generated by the nontrivial adjacency matrices has dimension 2. For a pair of 3-dimensional matrices lying in any such 2-dimensional ternary algebra, we have worked out in [12, Section 5] a number of techniques to determine if it is invertible or not and if so to compute explicitly the inverse pairs. In the following subsections, we make use of these techniques to choose invertible pairs from some of these ternary algebras and then obtain the corresponding 3D transforms according to Definition 2.2.1. It is not necessary that in the transform according to Definition 2.2.1 the matrix  $A$  should lie in the ternary subalgebra: it can lie in the entire ternary algebra.

#### 3.2 3D Transforms using the Projective Special Linear Group

In [41, Theorem 4.1], we have proved that given any 2-transitive permutation group  $G$  acting on a set  $\Omega$ , the orbits of  $G$  in its natural action on  $\Omega \times \Omega \times \Omega$  form the relations of an *AST*. Using this result, many examples of *AST*'s can be constructed by taking various 2-transitive permutation groups. In particular, if one takes  $G$  to be the group  $PSL(2, q)$  in its action on the projective line of  $q + 1$  points, then one has an *AST* which has precisely two non-trivial relations if  $q$  is odd, and only one non-trivial relation if  $q$  is even. Thus, if  $q$  is odd, then for the ternary algebra arising from the *AST* constructed in this manner, one has that the corresponding subalgebra  $\mathcal{F}$  is of dimension 2. We have proved:

**Proposition 3.2.1** *For a pair  $(A, B)$  of matrices given by*

$$A = A_4 + aA_5, \quad B = bA_4 + cA_4 \quad (22)$$

*with  $b \neq 0$ , one has that in the subalgebra  $\mathcal{F}$  for the *AST* constructed from  $PSL(2, q)$  where  $q$  is odd, an inverse pair  $(C, D)$  where*



$$C = A_4 + \alpha A_5, \quad D = \beta A_4 + \gamma A_5 \quad (23)$$

may be determined as follows:

(i) If  $q = 4t + 1$ , assume that  $a \neq -1$  and  $b \neq -c$ . Then, one has that  $\beta = (4t^2)^{-1}$  and  $\alpha, \gamma$  must satisfy the following non-linear equation:

$$(4t^2b)\gamma(1 + \alpha) + \alpha = -(ab + ac + c)(1 + a)^{-1}(b + c)^{-1}. \quad (24)$$

In particular, if for example,  $\alpha = 0$ , then an inverse pair is given by

$$C = A_4, \quad D = (4t^2b)^{-1}[A_4 - (ab + ac + c)(1 + a)^{-1}(b + c)^{-1}A_5] \quad (25)$$

(ii) If  $q = 4t + 3$ , assume that  $b = -ac$  and that  $(b + c + ab) \neq 0$ . Let

$$u := ac(b + c + ab)^{-1}, \quad v := (2t + 1)^{-2}(b + c + ab)^{-1} \quad (26)$$

Then

$$\beta = v(1 - u)^{-1} \quad (27)$$

provided  $u \neq 1$ , and  $\alpha, \gamma$  must satisfy the linear equation

$$\gamma + \alpha\beta = (2t + 1)^{-2}b^{-1} - 2\beta. \quad (28)$$

In some special cases, it is much easier to compute inverse pairs than the method described in 3.2.1. It follows from 3.2.1 that we have a plenty of choice for selecting invertible pairs in the ternary algebra arising from  $PSL(2, q)$ . For example, in 3.2.1 when  $q = 4t + 1$  choosing  $A$  and  $B$  such that  $b = 1, c = 0$  would readily give the pair  $(A_4 + aA_5, A_5)$  which is invertible and whose inverse pair is easily seen to be

$$C = A_4, \quad D = \frac{1}{4t^2} \left\{ A_4 - \frac{a}{1 + a} A_5 \right\} \quad (29)$$

Here, the adjacency matrices of the AST are of size  $v \times v \times v$  where  $v = q + 1 = 4t + 2$  for any integer  $t \geq 1$ . This gives us a transform for any matrix in the ternary subalgebra. Notice that one can construct explicitly the matrices  $(P, Q)$  for each choice of  $t$ . Note that here we take  $P = A_4 + aA_5$ ,  $Q = A_5$ .

### 3.3 3D Transforms from Block Designs

A 2-design  $(X, \mathcal{B})$  with the parameters  $b, v, k, \lambda$  is a family of  $k$ -subsets of a set  $X$  which are called *blocks* such that any 2-subset of  $X$  lies in exactly  $\lambda$  blocks,  $|X| = v$ ,  $|B| = b$ . In [41, Theorem 3.1], we showed that for a 2-design with  $\lambda = 1$  and  $k \geq 4$ , one can construct an AST which has precisely two non-trivial relations  $R_4$  and  $R_5$  where  $R_4$  consists of all triples  $(x, y, z)$  such that  $\{x, y, z\}$  lies in some block of  $\mathcal{B}$  and  $R_5$  consists of all triples  $(x, y, z)$  such that  $\{x, y, z\}$  does not lie in any block of  $\mathcal{B}$ . So, for this AST one has that the corresponding subalgebra  $\mathcal{F}$  has dimension 2 and we have developed a method to compute inverse pairs which is similar to the proof of Proposition 4.2.1. It is interesting to note that in this case  $\mathcal{F}$  is commutative.

### 3.4 3D Transforms from Affine groups

We have obtained the following result on how to construct an AST from the affine group.

**Proposition 3.4.1** *Consider the action of the affine group  $AGL(1, q)$  acting as a 2-transitive permutation group on the finite field  $GF(q)$ . Then, one obtains an AST with  $q + 2$  relations consisting of the orbits of  $AGL(1, q)$  on  $\Omega \times \Omega \times \Omega$  where  $\Omega = GF(q)$ . Here, each non-trivial relation has a unique (orbit) representative of the form*

$$(0, 1, d), \quad d \in GF(q) \setminus \{0, 1\} \quad (30)$$

If  $A^a$  denotes the adjacency matrix of the relation containing the element  $(0, 1, a)$  for any  $a \in GF(q) \setminus \{0, 1\}$ , then for  $a, b, c \in GF(q) \setminus \{0, 1\}$ , we have that

$$A^a A^b A^c = \begin{cases} A^{bc} & \text{if } c = a(a + b - 1)^{-1}, bc \neq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Further, if}$$

$b, c \in GF(q) \setminus \{0, 1\}$  such that  $bc = 1$ , then

$$I_1 A^b A^c = I_1, \quad A^b I_2 A^c = I_2, \quad A^b A^c I_3 = I_3$$

and if  $bc \neq 1$ , then all the three ternary products above are equal to 0.

When  $q = 5$ , then it follows from 4.4.1 that the AST thus obtained has exactly three nontrivial relations and the corresponding adjacency matrices are denoted in the above notation by  $A^2, A^3, A^4$  and let  $\mathcal{F}$  be the ternary subalgebra generated by these matrices (note that here  $A^3$  is different from the product  $AAA$ ). In the following result we give an explicit expression to compute inverse pairs in this particular subalgebra.

**Proposition 3.4.2.** *Consider the ternary subalgebra  $\mathcal{F}$  spanned by the nontrivial relation matrices in the AST generated by the action of  $AGL(1, 5)$  on  $GF(5)$ . A factor pair  $(A, B)$  in  $\mathcal{F}$  has an inverse pair  $(C, D)$  in  $\mathcal{F}$  only in the following two cases:*

Case 1. If

$$A = a_2 A^2 + a_4 A^4, \quad B = b_3 A^3 + b_4 A^4 \quad (31)$$

then the (left) factor pair  $(A, B)$  has an inverse pair  $(C, D)$  if and only if  $a_2, a_4, b_3, b_4$  are all nonzero, and then one has that

$$C = k(1/a_2 A^2 + 1/a_4 A^3), \quad D = (1/k)(1/b_3 A^2 + 1/b_4 A^4) \quad (32)$$

where  $k$  is an arbitrary nonzero number.

Case 2. If

$$A = a_2 A^2 + a_3 A^3, \quad B = b_2 A^2 + b_4 A^4 \quad (33)$$

then  $(A, B)$  has an inverse pair  $(C, D)$  if and only if  $a_2, a_3, b_2, b_4$  are all nonzero, and then

$$C = k(1/a_2 A^2 + 1/a_3 A^4), \quad D = (1/k)(1/b_2 A^3 + 1/b_4 A^4) \quad (34)$$

where  $k$  is an arbitrary nonzero number.

## 4 Uncertain Information Processing

Representation of data by higher dimensional arrays is of significant practical importance. In a large number of scientific areas we need to store data (e.g., signals, images, or statistical samples) in the form of arrays. In the classical approach we consider two-dimensional arrays only which are usually referred to as the *contingency tables* or simply as *matrices*. A number of properties of the data could be investigated in terms of the structure of the linear algebra generated by these matrices. However, we could also store data instead in the form of multi-dimensional arrays: one reason for doing this lies in storing the large data in a compact way. This kind of representation has been found to be useful in many applications especially in the processing of data in satellite reconnaissance photographs, medical imagery including X-ray images, image reconstruction, seismic records, and electron micrographs. There are applications of multi-dimensional arrays in other areas such as the design of experiments, transportation planning and defense strategic planning [13]. In the design of VLSI structures and architectures for parallel computers (e.g., a "Hypercube"), multi-dimensional arrays play a significant role.

Dempster-Shafer theory [17] for belief and plausibility measures is a well known area in knowledge engineering which has found many practical applications, e.g. in decision estimation, evaluation of software prototypes, medical diagnosis, to name just a few. There have been many significant further developments to this theory, and its relationship with the theory of approximate reasoning and fuzzy logic, e.g., [18]-[21], [25]-[27]. In a series of papers Yager ([22]-[24]) developed a reasoning system based on the *possibility-probability granule* amalgamating the well known Dempster-Shafer theory and the theory of approximate reasoning. In [22] this is developed by introducing transition matrices to represent the relationship between variables. The aim of this paper is to attempt to extend the work of Yager in a multi-dimensional setting. A multi-dimensional array, called a *transition matrix*, is defined which stores the joint probabilities of the occurrences of a set of  $n$  variables taking their values in different sets. Using the transition matrix we show that it is possible to compute the information regarding the probability of occurrences of the variables as certain matrix products. The multi-dimensional approach thus provides a framework to represent and analyze uncertain data in a compact way.

Let  $A, B, X$  be sets with finite cardinalities. Let  $A : X \rightarrow [0, 1]$  and  $B : X \rightarrow [0, 1]$ ,  $B^-(x) := 1 - B(x)$ . If  $V$  is a variable which takes values in  $X$ , we say that

$V$  is  $A$ .

to indicate that the value taken by  $V$  is known to be a member of  $A$ . Two measures have been obtained to examine if  $V$  is also in  $B$ ; these are called the *possibility measure*  $\text{poss}[B/A]$ , and the *certainty measure*  $\text{cert}[B/A]$  which are defined as follows:

$$\text{poss}[B/A] := \max_{x \in X} [B(x) \wedge A(x)] \quad (35)$$

$$\text{cert}[B/A] := 1 - \text{poss}[B^-/A] \quad (36)$$

We remark that  $\text{poss}[B/A]$  measures the degree of overlap of  $A$  and  $B$ , and  $\text{cert}[B/A]$  measures the degree to which  $B$  contains  $A$ . Further, if  $A$  and  $B$  are *crisp* sets (i.e., correspond to actual subsets of  $X$  indicated by characteristic functions), then we see that

$$\text{poss}[B/A] = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

$$\text{cert}[B/A] = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases} \quad (38)$$

Let  $\mathcal{P}(X)$  denote the set of all subsets of a finite set  $X$ . Let  $m : \mathcal{P}(X) \rightarrow [0, 1]$  such that

$$\sum_{A \in \mathcal{P}(X)} m(A) = 1 \quad (39)$$

The map  $m$  is called the *basic probability assignment function* (bpa). If  $A \in \mathcal{P}(X)$  and  $m(A) \neq 0$ , then  $A$  is called a *focal element* of  $m$ . Let  $B : X \rightarrow [0, 1]$ . Then  $Pl(B)$ , the *plausibility* of  $B$  is defined as

$$Pl(B) = \sum_i \text{Poss}(B/A_i) * m(A_i) \quad (40)$$

and  $Bel(B)$ , the *belief* of  $B$ , is defined as

$$Bel(B) = \sum_i (\text{cert}(B/A_i) * m(A_i)). \quad (41)$$

If  $m$  is a basic probability assignment function (bpa) from  $\mathcal{P}(X)$  then we say that

$V$  is  $m$ .

Such a proposition is called a *possibility-probability granule* ( $P - P$ -granule). A  $P - P$ -granule is a means of representing uncertain information in a convenient way and this has been studied extensively in [24] where a methodology has been developed to perform various operations on the  $P - P$  granules and implement inferential reasoning.

#### 4.1 Information Representation By Multi-dimensional Arrays

- The work by us reported here has been now published in the **IEEE Trans. Systems, Man & Cybernetics**, January, 1994.

In [23] two-dimensional arrays have been used to represent the relationships between variables. We have extended this approach to multi-dimensional arrays. For the sake of convenience we shall give here the extension only to three-dimensional arrays from where it could be extended in a straightforward manner to the  $n$ -dimensional case for an arbitrary  $n$  and we shall leave the details.

Let  $U, V, W$  be three variables taking their values in the sets

$$X = \{x_1, \dots, x_n\}, Y = \{y_1, \dots, y_n\}, Z = \{z_1, \dots, z_n\} \quad (42)$$

respectively. Let  $T$  be an  $n \times n \times n$  matrix with the property that for all  $1 \leq k \leq n$ ,  $1 \leq j \leq n$ ,  $1 \leq i \leq n$

$$\text{pr}[U \text{ is } z_k, V \text{ is } y_j, W \text{ is } x_i] = (T)_{ijk} \quad (43)$$

where  $\text{pr}$  denotes the probability. The 3-dimensional matrix  $T$  will be called the *transition matrix* of  $U, V, W$ .

For any permutation  $\sigma$  of the set  $\{1, 2, 3\}$ , let  $T^\sigma$  denote the  $n \times n \times n$  matrix whose  $(i, j, k)$  entry is equal to  $(T)_{\sigma(i)\sigma(j)\sigma(k)}$ . Since

$$\text{pr}[U \text{ is } z_k, V \text{ is } y_j, W \text{ is } x_i] = \text{pr}[V \text{ is } y_j, U \text{ is } z_k, W \text{ is } x_i] \text{ etc.} \quad (44)$$

it follows that  $T^\sigma = T$ , i.e.,  $T$  is symmetric with respect to all permutations of  $\{1, 2, 3\}$ . For example, we have  $(T)_{123} = (T)_{213}$  etc. Further,

$$T_{..i} = T_{i..} = T_{.i.} \quad (45)$$

It is possible to extend easily the definition of the transition matrix to the case when  $X, Y, Z$  do not have the same cardinalities and then to generalize some of our results which follow; we omit the details.

We now interpret some special *submatrices* of the transition matrix  $T$ . Consider  $T_{i..}$  for a fixed  $i$  such that  $1 \leq i \leq n$ . This  $n \times n$  submatrix consists of the  $i^{\text{th}}$  vertical layer of  $T$  showing all the values of probabilities where  $W$  is  $x_i$  and  $U, V$  taking all possible values. We have for fixed  $i, j$

$$\sum_{k=1}^n (T)_{ijk} = 1 \quad (46)$$

using the standard laws of probability. Thus, in the  $i^{\text{th}}$  layer, the sum of entries on the  $j^{\text{th}}$  row is equal to 1. Again, consider  $T_{.jk}$ . This represents a "line" in the 3-dimensional matrix  $T$ . We have for fixed  $j, k$

$$\sum_{i=1}^n (T)_{ijk} = 1 \quad (47)$$

Similarly, we have for fixed  $i, k$

$$\sum_{j=1}^n (T)_{ijk} = 1 \quad (48)$$

The corresponding entries in the above summation come from the matrix  $T_{i..k}$  which consists of the  $k^{\text{th}}$  column of the  $i^{\text{th}}$  vertical layer of  $T$ .

We now show how the transition matrix  $T$  could be used to obtain expressions for various kinds of uncertainties. These results show the significant role played by the multi-dimensional transition matrices.

**Theorem 4.1.1.** Let  $P, Q, R$  be three  $1 \times n$  matrices whose  $(1, j)$  entries are defined to be  $\text{Pr}(W \text{ is } x_j)$ ,  $\text{Pr}(V \text{ is } y_j)$  and  $\text{Pr}(U \text{ is } z_j)$  respectively. Then, the knowledge of any two of the matrices  $P, Q, R$  is enough to capture the third matrix in a unique manner by using the information regarding the transition matrix  $T$ . More specifically, we have for  $1 \leq i \leq n$

$$(P)_{1i} = Q \times (R \times T_{..i})^t = R \times (Q \times T_{..i})^t \quad (49)$$

$$(Q)_{1i} = P \times (R \times T_{..i})^t = R \times (P \times T_{..i})^t \quad (50)$$

$$(R)_{1i} = P \times (Q \times T_{..i})^t = Q \times (P \times T_{..i})^t \quad (51)$$

where  $^t$  denotes the transpose of a matrix in the usual sense.

The first of the above three shows an interesting symmetry in the sense that  $Q$  and  $R$  can be interchanged. Similar symmetry also holds in the other two equations of the above set.

**Corollary 4.1.2:**

$$P_{1j} = (R \times T_{..j}) \times Q^t = (Q \times T_{..j}) \times R^t \quad (52)$$

Also, we have similar extensions of the second and third equations in Theorem 5.1.1.

**Example 4.1.3:** Let  $X = \{x_1, x_2, x_3\}$ ,  $Y = \{y_1, y_2, y_3\}$  and  $Z = \{z_1, z_2, z_3\}$ . In the notation of Theorem 5.1.1, let

$$P = \begin{bmatrix} .1 & .5 & .4 \end{bmatrix}, \quad Q = \begin{bmatrix} .2 & .5 & .4 \end{bmatrix} \quad (53)$$

Let the transition matrix  $T$  be given by

$$T_{1..} = \begin{bmatrix} 0.3 & 0.1 & 0.6 \\ 0.1 & 0.7 & 0.2 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}, \quad T_{2..} = \begin{bmatrix} 0.1 & 0.7 & 0.2 \\ 0.7 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \quad T_{3..} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.9 & 0.6 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} \quad (54)$$

We observe that the 3-dimensional matrix  $T$  is symmetric with respect to all possible permutations of  $\{1, 2, 3\}$ , as remarked earlier. We now compute the matrix  $R$  from  $P$ ,  $Q$  and  $T$ . We have

$$(R)_{11} = P \times (Q \times T_{..1}^t) = \begin{bmatrix} .1 & .5 & .4 \end{bmatrix} \times \left( \begin{bmatrix} .2 & .6 & .4 \end{bmatrix} \times \begin{bmatrix} .3 & .1 & .6 \\ .1 & .7 & .2 \\ .6 & .2 & .2 \end{bmatrix} \right)^t = 0.376 \quad (55)$$

By computing  $(R)_{22}, (R)_{33}$  in a similar manner, we get

$$R = \begin{bmatrix} .376 & .28 & .344 \end{bmatrix} \quad (56)$$

Notice that the sum of the entries of  $R$  is equal to 1, as is expected since they represent probabilities.

We observe that if we had used the identity,  $(R)_{1j} = Q \times (P \times T_{..j})^t$ , then the value of  $R$  we get is the same as the one obtained above. Now, we use the value of  $R$  obtained already, together with the given values of  $P$  and  $T$  to calculate the matrix  $Q$ . We get

$$(Q)_{11} = P \times (R \times T_{..1}^t) = \begin{bmatrix} .1 & .5 & .4 \end{bmatrix} \times \left( \begin{bmatrix} .376 & .28 & .344 \end{bmatrix} \times \begin{bmatrix} .3 & .1 & .6 \\ .1 & .7 & .2 \\ .6 & .2 & .2 \end{bmatrix} \right)^t = 0.32608 \quad (57)$$

Similarly, after calculating the remaining entries of  $Q$ , we obtain that

$$Q = \begin{bmatrix} .32608 & .34528 & .32864 \end{bmatrix} \quad (58)$$

Comparing the value of  $Q$  obtained in (58) with the value of  $Q$  given already, we observe that they are not equal. Thus, the above analysis shows an interesting phenomenon, namely, that the relationship between the matrices  $P, Q, R$  is *nonlinear*. Thus we have:

**Remark 4.1.4.** Given  $P$  and  $R$ , the value of  $Q$  which can be computed from the transition matrix  $T$ , is not necessarily such that it, together with  $P$  (respectively  $R$ ), would yield again the given value of  $R$  (respectively  $P$ ) when we use the corresponding equation in the statement of the Theorem 4.1.1.

## 4.2 Conclusion of this section

Multidimensional arrays have extensive practical applications in a large number of other areas. However, to our knowledge, no work has been done so far to develop the Dempster-Shafer theory in a multidimensional setting. Here, we have initiated such a study and shown how to manipulate uncertain information using a multi-dimensional array called the transition matrix. It is hoped that this correspondence would stimulate further research on multidimensional processing of uncertain information.

## 5 Miscellaneous Topics – Summaries

We give in this section summaries of some other works done during the tenure of the grant.

### 5.1 Parallel Image Processing

The single-instruction-multiple-data (SIMD) mode of computing is considered to be the best mode for parallel processing in optics and there are several existing parallel optical processors of the SIMD type including the OPALS [29], DOLCIP [30] and SSP [31]. A comprehensive language called the image-logic algebra (ILA) has been introduced [32] for optical computing in the SIMD mode, its system architecture and optical implementation have also been considered [32]. Furthermore, several algorithms in the ILA have been developed [33] for diverse tasks such as the computer-aided design (CAD) and numerical data processing – these show the power and versatility of the ILA language in handling a wide range of problems. Our objective is to explore further applications of the ILA to do parallel optical image processing. It has been already observed [32]-[33] that the principal operations of mathematical morphology and binary image algebra can be described by the ILA. We show that only a few of the basic operations of the ILA are needed to develop a number of parallel algorithms for image processing if we use an algebraic method for processing images. We have shown that the optical parallel processing of binary images can be developed in the single-instruction-multiple-data (SIMD) generic language called the image-logic algebra (ILA), using the polygonal approach for representing images.

- This work has been accepted for publication as a single-author paper in the journal **Applied Optics**.

### 5.2 Permutation Representation By Trees

Permutations arise naturally in connection with problems in many fields such as data encryption, parallel processing, computer networking and computational algebra. Extensive work has been done in the past to represent and generate permutations (see e.g., Lehmer [39] and Sedgewick [?] for surveys). It is of considerable interest to store and retrieve permutations (and combinations) in an efficient manner in the computer memory. Recently, Arnow [36] developed an interesting method to store permutations, combinations and dihedral elements in a “tree”-structure, and gave some algorithms to generate the nodes and traverse these trees. We have developed algorithms to represent permutations and combinations in the *lexicographic order*. Lexicographic (or dictionary) order has numerous practical applications — in discrete mathematics (e.g., kn:Graham), in data structures (e.g., kn:Aho, kn:Gonnet), and in parallel processing and networking (e.g., kn:Akl). In

Liu kn:Liu, algorithms to generate permutations in lexicographic order have been described. However, he does not address the problems of storing and retrieving permutations which are certainly problems of considerable practical interest. The main contribution of my work is to explore these problems using the tree data structure. Our algorithms can be implemented using any programming language which is capable of recursive calls.

- Our paper has been accepted for publication in the journal **Computers & Math. Applications**.

### 5.3 3D Polygonal Arcs

During an extended visit to the Center for Automation Research, University of Maryland, College Park, the P.I. collaborated with Professor Azriel Rosenfeld to investigate geometric properties of 3D polygonal arcs.

A *polygonal arc* in  $d$  dimensions is a geometrical figure having  $n$  sides defined by a sequence of  $(n + 1)$   $d$ -tuples of (real) coordinates representing the vertices, successive vertices are the endpoints of a common side. A polygonal arc is called a (closed) *polygon* if the last vertex coincides with the first vertex. Polygonal arcs and polygons are of significant interest in robot vision and arise in a number of problems such as the analysis of the edges of solids, matching, and path planning. Various aspects of the case  $d = 2$  have been investigated by a number of authors. We investigate properties polygonal arcs and polygons with special emphasis to the case  $d = 3$ . We assume that all the vertices of a polygonal arc (or polygon) that we shall study in this paper, lie inside a bounded region. One method to represent a polygonal arc (up to translation and rotation) is by the lengths of its  $n$  sides and the  $n - 1$  angles between successive sides. This representation is convenient, for example, when the sides of the polygonal arc are parallel to the coordinate axes: such arcs are called *isothetic arcs*. In [41] we have investigated 2 dimensional isothetic arcs where the angles between consecutive sides are specified compactly by single bits indicating whether the angle corresponds to a *left turn* or a *right turn*. (For a closed polygon there is also an angle between the first and last sides, but it is redundant, and we can tell from the  $n$  lengths and  $n - 1$  angles whether or not the polygon is closed.)

Given a polygonal arc in an arbitrary dimension  $d$ , one sees that some geometrical and topological properties can be examined. Similarly, one can decide about the intersection of two (or more) polygonal arcs or polygons. In the case of dimension 2 (only), one has that a closed non-selfintersecting polygon divides the plane into an inside and an outside, and also we can decide if one of two polygons surrounds the other. When the dimension of the space is 3, a more interesting issue arises: to determine if a closed polygon is knotted, or if two closed polygons are linked. We study these questions. Our solution to the problem makes use of homotopy groups computed from so-called *Wirtinger* projection(s) of the polygon(s) onto a plane.

- This joint work with Prof. A. Rosenfeld has now been published in June, 1994 in the **J. Visual Communication and Image Representation**.



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## 6 List of Papers Written Under the Project

1. P. Bhattacharya and N.P. Mukherjee, "A Multi-dimensional Approach to the Representation of Uncertain Information", *IEEE Trans. on Systems, Man & Cybernetics*, vol. 24, no. 1, pp. 107-111 (1994).
  2. D.M. Mesner and P. Bhattacharya, "A Ternary Algebra arising from Association Schemes on Triples", *Journal of Algebra* (Publisher: Academic Press), vol. 164, no. 1, pp. 595-613, (1994).
  3. P. Bhattacharya and A. Rosenfeld, "Polygons in Three Dimensions", *J. Visual Communication & Image Representation*, ivol. 5, pp. 139-147 (1994).
  4. P. Bhattacharya, "Parallel, Optical Image Processing by the Image-Logic Algebra", *Applied Optics* (Publisher: Optical Society of America), accepted for publication scheduled to appear in vol. 33.
  5. P. Bhattacharya, "On The Representation of Permutations By Trees", *Computers & Math. Applications* (Publisher: Pergamon Press), accepted for publication.
  6. P. Bhattacharya and S. Hungenhally, "A Mathematical Framework for Visual Receptive Fields Using Discriminant Functions", *Proc. IEEE Internat. Conf. on Systems, Man & Cybernetics*, San Antonio, Texas, October, 1994, accepted.
  7. P. Bhattacharya, "A New Three Dimensional Transform Using A Ternary Product", submitted to *IEEE Trans. Signal Processing*, currently under review.
  8. P. Bhattacharya, "A Three-Dimensional Transform Using Ternary Products and Inverse Pairs", *Proc. Internat. Congress of Math.*, accepted. Zürich, Switzerland, August, 1994.
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